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Unoriented Strings, Loop Equations, and $\mathcal{N} = 1$ Superpotentials from Matrix Models

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Abstract

We apply the proposal of Dijkgraaf and Vafa to analyze $\mathcal{N} = 1$ gauge theory with $SO(N)$ and $Sp(N)$ gauge groups with arbitrary tree-level superpotentials using matrix model techniques. We derive the planar and leading non-planar contributions to the large M $SO(M)$ and $Sp(M)$ matrix model free energy by applying the technology of higher-genus loop equations and by straightforward diagrammatics. The loop equations suggest that the $\mathbb{RP}^2$ free energy is given as a derivative of the sphere contribution, a relation which we verify diagrammatically. With a refinement of the proposal of Dijkgraaf and Vafa for the effective superpotential, we find agreement with field theory expectations.

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1 Introduction

Recently Dijkgraaf and Vafa [1–3] have proposed that the exact low-energy superpotential of certain $\mathcal{N} = 1$ supersymmetric gauge theories is captured by the large M behavior of certain associated $M \times M$ matrix models. This is quite remarkable, as it reduces the problem of what is, in general, strongly-coupled physics of the confining phase of pure gauge theory to the zero-dimensional dynamics of a matrix integral. Furthermore, the gauge theory quantities are computed from just the planar graphs of the matrix theory, nevertheless capturing finite N results in $SU(N)$ gauge theory.

The conjecture was initially tested for $\mathcal{N} = 2$ $SU(N)$ gauge theories softly broken to $\mathcal{N} = 1$ by a tree-level superpotential for the adjoint chiral superfields [1] and for the $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ $SU(N)$ SYM [3–5]. The conjecture has since been extended to a number of other cases [2,6–26] and has been derived from $SU(N)$ gauge theory [27–29].

In this work we use matrix techniques to analyse $\mathcal{N} = 1$ gauge theory with $SO(N)$ and $Sp(N)$ gauge groups. By a careful consideration of the planar and leading non-planar corrections to the large M $SO(M)$ and $Sp(M)$ matrix models, we derive the matrix model free energy. We do this both by applying the technology of higher-genus loop equations of [30,31] and by straightforward diagrammatics (see *e.g.* [32,33]).

The outline of the paper is as follows. In section 2, we discuss general features of the four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry softly broken to $\mathcal{N} = 1$. We also suggest a result for the superpotential of the $\mathcal{N} = 1^*$ theory with gauge group $SO(2N)$ which is based on a generalization of a derivation of Dorey [34] for $SU(N)$ gauge group. In section 3, we discuss the geometric engineering of the softly broken $\mathcal{N} = 2$ gauge theories by wrapping D5-branes and O5-planes on compact cycles of generalized conifolds.

The corresponding matrix models are introduced and solved in the large M limit in section 4. As for $SU(M)$, we find that the loop equation for the resolvent of the matrix model describes a Riemann surface which is identified with a factorization of the spectral curve of the $\mathcal{N} = 2$ gauge theory. The large M solution of the matrix models computes the special geometry of Type IIB string theory on the associated Calabi-Yau manifold.

In section 5, we discuss the application of the higher-genus loop equations to the computation of the $\mathbb{RP}^2$ contribution to the free energy. The loop equations take the form of integral equations which give recursion relations between the contributions to the resolvent at each genus. They suggest a very simple solution for the $\mathbb{RP}^2$ contribution in terms of the sphere contribution. In fact, the one crosscap contribution to the resolvent ω_1 satisfies

$$\omega_1 = \pm q \frac{\partial \omega_0}{\partial S_0}, \tag{1.1}$$

where ω_0 is the contribution to the resolvent from the sphere. We verify this relationship by explicitly enumerating several types of diagrams. We find that the contribution to the free energy $\mathcal{F}_1$ from $\mathbb{RP}^2$ and $\mathcal{F}_0$ from S^2 are related by

$$\mathcal{F}_1 = \pm q \frac{\partial \mathcal{F}_0}{\partial S_0}, \tag{1.2}$$

where S_0 is half of the 't Hooft coupling for the SO/Sp component of the matrix group. We determine the proportionality constant q from the diagrammatics to be $q = \frac{g_s}{4}$.

Our results suggest a refinement of the proposal of Dijkgraaf and Vafa for the effective superpotential in the case of SO and Sp gauge groups. We find that

$$W_{\text{eff}} = Q_{D5}\, \partial \mathcal{F}_0 \partial S + Q_{O5}\, \mathcal{G}_0 - 2\pi i\, \tau\, S, \tag{1.3}$$

where Q_{D5} is the total charge of D5-branes, Q_{O5} is the total charge of O5-planes, $\mathcal{F}_0$ is the contribution to the matrix model free energy from diagrams with the topology of a sphere and $\mathcal{G}_0$ is proportional to $\mathcal{F}_1$, the contribution to the free energy from $\mathbb{RP}^2$ diagrams. We use (1.3) to obtain results consistent with gauge theory expectations. In particular, the matrix model is consistent with the requirement that there is a degeneracy of the massive vacua of the gauge theory given by h, the dual Coxeter number of the gauge group. In the case of the $\mathcal{N} = 1^*$ $SO(2N)$ theory, which we discuss in section 6.2, we find that the critical value of the superpotential exactly matches the result obtained from our gauge theory arguments in section 2. We end with a discussion of our results and point out several areas for future development. Various supporting technical calculations are contained in appendices.

In the course of this work, two papers on matrix models with SO/Sp groups have appeared. In [10], aspects of the geometric engineering of the gauge theories, as well the leading order in M computation of the free energy of quartic orthogonal and symplectic matrix ensembles are discussed. We use a different basis of matrices and we account for the appearance of diagrams involving pairs of twisted propagators that are not including in the oriented theory. More recently, while this manuscript was in a final stage, [35] appeared. These authors discuss a perturbative derivation of the matrix model along the lines of [28], including a discussion of $\mathbb{RP}^2$ corrections. Their results confirm aspects of the refinement (1.3) that we found was necessary for the computation of the gauge theory effective superpotential.

2 Results from $\mathcal{N}=1$ Gauge Theories

In this section we review some results about $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetric gauge theories, specifically the spectral curves and how they factorize when $\mathcal{N}=2$ is softly broken to $\mathcal{N}=1$. We focus on the case of SO and Sp gauge groups (SU was treated in [36], which this discussion follows).

2.1 $\mathcal{N}=2$ Softly Broken to $\mathcal{N}=1$

As is well known [37, 38][1], the moduli space of $\mathcal{N}=2$ gauge theories is governed by a "spectral curve", the periods of which give the masses of BPS objects in the theory (W-bosons, monopoles and dyons). In [41–43] these spectral curves were found for the SO/Sp gauge groups. For a rank-r gauge theory, the spectral curve is a genus r hyperelliptic curve,

$$y^2 = P_{2r+2}(x, \{\phi_i\}), \tag{2.1}$$

where P_{2r+2} is a polynomial of degree $2r+2$ in the x that also depends on the moduli ϕ_i. The SO and Sp spectral curves can also be written as a genus $2r-1$ curve,

$$y^2 = P_{2r}(x^2, \{\phi_i\}), \tag{2.2}$$

which is therefore symmetric under the $\mathbb{Z}_2$ action $x \mapsto -x$ and is a double cover of the genus N curve (2.1) via this map.

In $\mathcal{N}=1$ language, the $\mathcal{N}=2$ vector multiplet of the Yang-Mills theory is decomposed into an adjoint chiral superfield Φ and an $\mathcal{N}=1$ vector superfield V. $\mathcal{N}=2$ supersymmetry can be broken to $\mathcal{N}=1$ by an appropriate gauge-invariant superpotential term for Φ. Because the trace of odd powers of matrices in the Lie algebra of $SO(N)/Sp(N)$ vanishes, in contrast to the $U(N)$ case discussed in [1–3], the superpotential deformation for $SO(N)/Sp(N)$ only includes polynomial terms of even

[1] See [39, 40] for reviews.

degree:

$$W_{\text{tree}}(\Phi) = \sum_{k=1}^{n+1} \frac{g_k}{2k} \text{Tr}(\Phi^{2k}). \tag{2.3}$$

A superpotential W_{tree} of order $2n+2$ breaks the gauge symmetry down to a direct product of $n+1$ subgroups, *e.g.*:

$$SO(N) \to SO(N_0) \times U(N_1) \times \ldots \times U(N_n), \tag{2.4}$$

where $N = N_0 + 2N_1 + \cdots + 2N_n$.

The $U(1)$ factors in this theory decouple in the IR. In the supersymmetric vacua of the $\mathcal{N}=1$ theory $r-n$ mutually local monopoles simultaneously become massless and condense, leading to confinement of the gauge theory [36]. The condition that $r-n$ mutually local monopoles become massless leads to a "factorization locus" in the moduli space of the spectral curve, where $r-n$ of the (non-intersecting) cycles of the spectral curve are simultaneously pinching off to zero-volume.

Imposing this condition is therefore equivalent to the factorization [44]

$$y^2 = \prod_{i=1}^{r-n} (x^2 - p_i^2)^2 \prod_{j=1}^{2n} (x^2 - q_i^2), \tag{2.5}$$

where $p_i \neq p_j, q_i \neq q_j$ for $i \neq j$. On this locus we then obtain the "reduced spectral curve"

$$y^2 = \prod_{j=1}^{2n} (x^2 - q_i^2), \tag{2.6}$$

which is a genus $2n-1$ curve. This curve parameterizes the $\mathcal{N}=2$ vacua that are not lifted by the deformation to $\mathcal{N}=1$ (2.3). Notice that the factorized curves now have a similar form for SO and Sp, and the curve is still invariant under $x \mapsto -x$ (this implies that the branch points come in pairs: $(-q_i, q_i)$). This reduced spectral curve will be derived from string theory in the following section by taking an orientifold action on the configuration of D-branes on a generalized conifold that engineers this $\mathcal{N}=1$ gauge theory.

The low-energy effective superpotential of these gauge theory can be obtained from the reduced spectral curve as discussed by [36, 44]. It will take the form

$$W_{\text{eff}} = \sum_i \left(\hat{N}_i \Pi_i - 2\pi i \tau_i S_i \right), \tag{2.7}$$

where $2\pi i S_i$ are the periods of the meromorphic 1-form $y\,dx$ around the A-cycles of the spectral curve, Π_i the corresponding periods around the B-cycles, and $\hat{N}_i$ is

$$\hat{N}_i = \begin{cases} N_i & SU(N_i), \\ \frac{N_i}{2} - 1 & SO(N_i), \\ N_i + 1 & Sp(N_i). \end{cases} \tag{2.8}$$

By contrast, we will find that the shift $N_i \mapsto \hat{N}_i$ emerges in the matrix model by considering the first subleading corrections to the large M expansion, coming from Feynman diagrams of topology $\mathbb{RP}^2$.

Recently the gaugino effective superpotential has been perturbatively derived from $SU(N)$ gauge theory [27–29]. It is interesting to note that similar arguments[2] to those of [28] can be used to argue that only diagrams with at most one boundary (if quark flavors are present) or crosscap will contribute to the gauge theory superpotential [3, 13, 16].

2.2 The $\mathcal{N} = 1^*$ Theories

The $\mathcal{N} = 1^*$ theories arise as deformations of $\mathcal{N} = 4$ Yang-Mills theory by mass terms for the three adjoint $\mathcal{N} = 1$ chiral fields. The total superpotential is

$$W = \operatorname{tr}\left(\Phi_1[\Phi_2, \Phi_3] + \sum_{i=1}^{3} m_i\, \Phi_i^2\right) \tag{2.9}$$

and the F-flatness conditions can be written as

$$[\Phi_i, \Phi_j] \propto i\epsilon_{ijk}\Phi_k. \tag{2.10}$$

Supersymmetric vacua are then obtained by embedding $SU(2)$ representations into the gauge group G. In particular, for $G = SU(N)$, the embeddings are classified by the divisors d of N, leading to $\sum_{d|N} d$ massive vacua [45].

Dorey [34] (see also [46]), following the approach of [47], compactified the theory with $SU(N)$ gauge group on a circle of radius R. The degrees of freedom of the effective $2+1$–dimensional theory are $r = \operatorname{rank}(G)$ complex Abelian scalar fields X_a which are composed of the Wilson lines and the scalars dual to the massless photons of the theory.

[2]We thank Jaume Gomis and Jongwon Park for discussions on this issue. The same observation about crosscap contributions was made in [35].

The moduli space of the theory is

$$\mathcal{M} = E_\tau^r / \mathcal{W}_G, \tag{2.11}$$

where E_τ is the elliptic curve parameterized by each X_a and $\mathcal{W}_G$ is the Weyl group of G.

By several arguments, including the relationship between the elliptic Calogero-Moser systems and the $\mathcal{N} = 2$ theories which can be softly broken to the $\mathcal{N} = 1^*$ theory, Dorey found that the superpotential of the $2+1$–dimensional theory took the form

$$W = c \sum_{a>b} \wp(X_a - X_b), \tag{2.12}$$

where $\wp(z)$ is the Weierstrass function. Dorey argued that the coefficient c was independent of the radius R, so that critical values of (2.12) (which depend on the modular parameter τ and not the X_a) could be evaluated in the vacua of the theory and extrapolated directly to the $R \to \infty$ limit.

It is interesting to ask what the generalization of (2.12) is to arbitrary gauge groups G. The modular properties of the superpotential that were crucial to Dorey's argument must still be preserved, so the superpotential should remain a sum of Weierstrass functions. An obvious guess for the argument of these functions is to replace $X_a - X_b$ by the sum over the positive roots $\sum_{\alpha>0} \alpha \cdot X$. The integrable systems approach [48, 49, 45, 50] to $\mathcal{N} = 2$ theories is a promising route to this result. In fact, D'Hoker and Phong [51–53] have determined the integrable systems that govern a large class of $\mathcal{N} = 2$ theories with gauge group G. An application of the techniques of [45,34] to the soft breaking of these theories to $\mathcal{N} = 1^*$ suggests that the correct superpotential for gauge group G is

$$W = c \left(\sum_{\{\alpha_L > 0\}} \wp(\alpha_L \cdot X) + \sum_{\{\alpha_S > 0\}} \wp_{\nu(\alpha_S)}(\alpha_S \cdot X) \right). \tag{2.13}$$

where $\alpha_{L,S}$ are the long and short positive roots of the Lie algebra of G, respectively, and $\wp_\nu(z)$ are the *twisted* Weierstrass functions

$$\wp_\nu(z) = \sum_{\sigma=0}^{\nu-1} \wp\left(z + 2\omega_a \frac{\sigma}{\nu} \right). \tag{2.14}$$

defined in [51]. For non-simply laced groups, roots of only two different lengths appear: $\nu(\alpha) = 1$ for all long roots, $\nu(\alpha) = 2$ for all short roots of $\mathfrak{b}_n$, $\mathfrak{c}_n$ $\mathfrak{f}_4$, while $\nu(\alpha) = 3$ for the short roots of $\mathfrak{g}_2$. It would be interesting to prove the result (2.13) [54].

For $SO(2N)$, since it is simply-laced, twisted Weierstrass functions do not appear, and we obtain

$$W = c \sum_{a>b} \left[\wp(X_a - X_b) + \wp(X_a + X_b)\right]. \tag{2.15}$$

Following [34], we can evaluate this in the k^{th} confining vacuum to find (up to an additive constant)

$$W \sim E_2\big((\tau + k)/(2N - 2)\big), \tag{2.16}$$

where $E_2(\tau)$ is the second regularized Eisenstein series. It is tempting to conjecture that the result for arbitrary G will take this form with the obvious substitution of h, the dual Coxeter number of G for $2N - 2$, but this remains to be verified.

3 Calabi-Yau Geometry

We now review the string theoretic engineering of a softly broken $\mathcal{N} = 2$ gauge theory with SO/Sp gauge group [44,36,55,1]. We consider type IIB string theory compactified on the non-compact A_1 fibration

$$u^2 + v^2 + w^2 + W'(x)^2 = 0, \tag{3.1}$$

where $W(x)$ is a degree $n + 1$ polynomial, which will later be related to the tree level superpotential. This fibration has singularities at the critical points of $W(x)$. In the neighborhood of those singularities, we can introduce the coordinate $x' = W'(x)$. Then it is easy to see that the singularities are all conifold singularities.

This generalized conifold can be de-singularized in two ways: it can be resolved or it can be deformed. The resolution is given by the surface

$$\begin{pmatrix} u + iv & w + iW'(x) \\ -w + iW'(x) & u - iv \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{3.2}$$

in $\mathbb{C}^4 \times \mathbb{P}^1$. In this geometry each singular point is replaced by a $\mathbb{P}^1$. These $\mathbb{P}^1$'s are disjoint, holomorphic, have the same volume and are homologically equivalent.

The latter property can be seen by making use of the fibration structure away from $W'(x) = 0$. This A_1 fibration over the x plane induces a fibration of non-holomorphic S^2's over the x plane. This S^2 cannot shrink to zero size as one approaches a critical point of W in the x plane, but it becomes the holomorphic $\mathbb{P}^1$ of the resolution.

We can now construct a softly broken $\mathcal{N} = 2$ $U(N)$ gauge theory with tree level superpotential $W(x)$ by wrapping N D5-branes around the S^2. This is an UV definition of the theory. A classical supersymmetric vacuum is obtained by minimizing the volume of the D5-branes. This amounts to distributing a collection of N_i D5-branes over the n minimal-volume holomorphic $\mathbb{P}^1$'s at the critical points of W. The $U(N)$ gauge symmetry is then spontaneously broken to $U(N_1) \times \cdots \times U(N_{n-1})$.

We now want to consider an orientifold of this theory[3]. Since we started with a type IIB theory on a Calabi Yau, we have to combine the worldsheet orientation reversal with a holomorphic involution of the Calabi-Yau (an anti-holomorphic involution would be appropriate for the IIA theory). Furthermore we want to fix one of the $\mathbb{P}^1$'s and act freely on the rest of the Calabi Yau geometry. This can be done if $W(x)$ is an even polynomial of order $2n$. In terms of the fibration structure of the Calabi-Yau this means that the critical points of $W'(x)$ come in pairs $(-x_i, x_i)$ and one critical point is fixed at $x_0 = 0$. Then

$$(u, v, w, x, \lambda_1, \lambda_2) \mapsto (-u, -v, -w, -x, \lambda_1, \lambda_2) \tag{3.3}$$

is a holomorphic involution of the geometry (3.2), which leaves only the $\mathbb{P}^1$ at $u = v = w = x = 0$ fixed. In the string theory this means that there is an O5-plane wrapping this $\mathbb{P}^1$ in the Calabi-Yau geometry.

There are essentially two choices of O5-plane with which we can wrap the fixed $\mathbb{P}^1$. They are distinguished by a different choice of worldsheet action and carry RR 5-form charge of ± 1 (the RR charge of an O$p^{\pm}$-plane is $\pm 2^{p-5}$ in conventions where we count the charge of $N/2$ D-branes but not their $N/2$ images). The orientifold contribution to the RR charge of objects wrapping the $\mathbb{P}^1$ will cause a shift in the coefficient N_0 in the flux-generated superpotential on the deformed Calabi-Yau geometry, as explained below.

Now we can construct a softly broken $\mathcal{N} = 2$ $SO(N)/Sp(N/2)$ gauge theory with tree level superpotential $W(x)$ by wrapping N D5-branes around the S^2 and then

[3] Orientifolds were discussed in the A-model in [56, 55, 57, 58], while the discussion of [59] is more closely related to the B-model which is our interest here.

performing the orientifold. The gauge symmetry is again broken $SO(N) \mapsto SO(N_0) \times U(N_1) \times \cdots \times U(N_{n-1})$ or $Sp(N/2) \mapsto Sp(N_0/2) \times U(N_1) \times \cdots \times U(N_{n-1})$ respectively with $N = N_0 + 2N_1 + \cdots + 2N_{n-1}$.

If we flow this ultraviolet theory to the infrared, there will be a confinement transition. In string theory this is described by a geometric transition in which the resolved conifold geometry with wrapped D5-branes and O5-planes is replaced by a deformed conifold geometry [60]

$$u^2 + v^2 + w^2 + W'(x)^2 - f(x) = 0, \tag{3.4}$$

where $f(x)$ is an even polynomial of degree $2n-2$. Such a polynomial represents the most general normalizable deformation of the singular conifold that still respects the holomorphic involution (3.3). For a reasonably small $f(x)$, each critical point of $W'(x)$ is replaced by two simple zeros of $W'(x)^2 - f(x)$. This means that each $\mathbb{P}^1_i$ is replaced by a 3-sphere A_i with 3-form RR-flux H through it, equal to the amount of D5-brane and O5-plane charge on the $\mathbb{P}^1_i$. The orientifold acts on one 3-sphere A_0 as the antipodal map, while the other 3-spheres are mapped to each other in pairs A_i and A_{-i}. Note that there is no orientifold fixed plane anymore.

The coefficients in $f(x)$ are normalizable modes that are localized close to the tip of the conifold. The coefficients in $f(x)$ are determined by the periods

$$S_i = \frac{1}{2\pi i}\int_{A_i} \Omega. \tag{3.5}$$

These periods S_i can be interpreted as the gaugino condensates of the gauge theory. There are non-compact 3-cycles B_i that are dual to the A_i. The periods of the B-cycles are

$$\frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{B_i} \Omega, \tag{3.6}$$

where $\mathcal{F}_0$ is the prepotential. One needs to introduce a cutoff in order to make these periods finite.

The flux through the cycles A_i is determined in terms of the RR-charges of the D-brane and O-plane configuration

$$\begin{aligned} N_0 \pm 2 &= \int_{A_0} H, \\ N_i &= \int_{A_i} H, \quad i \neq 0, \end{aligned} \tag{3.7}$$

and the flux through the cycles B_i is given in terms of the coupling constants

$$\tau_i = \int_{B_i} H. \tag{3.8}$$

Since there is no orientifold fixed plane, there are no contributions to the effective superpotential for the gaugino condensate from unoriented closed strings [57]. It is then given by the flux superpotential [61–64]

$$W_{eff}(S_i) = \int H \wedge \Omega, \tag{3.9}$$

where the integral is taken only over half of the covering space of the orientifold. Using the expressions for the periods and the fluxes and taking into account the orientifold projection, we get

$$W_{\text{eff}}(S_i) = \left(\frac{N_0}{2} \pm 1\right) \frac{\partial \mathcal{F}_0}{\partial S_0} + \sum_{i>0} N_i \frac{\partial \mathcal{F}_0}{\partial S_i} - \frac{1}{2}\tau_0 S_0 - \sum_{i>0} \tau_i S_i. \tag{3.10}$$

This result could also have been computed on the open string side before the transition. On the open string side there is no flux through any 3-cycles, so there is no contribution to the superpotential due to closed oriented strings. But there are two kinds of other contributions to the effective superpotential: the open string contributions (disc diagrams) and the contributions due to closed unoriented strings at the orientifold fixed plane ($\mathbb{RP}^2$ diagrams). The contribution due to the open strings is the equal to one half that of the theory without the orientifold, *i.e.*, it is

$$W^o_{\text{eff}}(S_i) = \frac{N_0}{2} \frac{\partial \mathcal{F}_0}{\partial S_0} + \sum_{i>0} N_i \frac{\partial \mathcal{F}_0}{\partial S_i} - \frac{1}{2}\tau_0 S_0 - \sum_{i>0} \tau_i S_i. \tag{3.11}$$

The contribution due to the unoriented closed strings then must be

$$W^u_{\text{eff}}(S_i) = W_{eff}(S_i) - W^o_{eff}(S_i) = \pm \frac{\partial \mathcal{F}_0}{\partial S_0}. \tag{3.12}$$

We will confirm this result in our matrix model computation.

4 The Classical Loop Equation

We first consider the saddle point evaluation of the one matrix integral for $SO(M)$ or $Sp(M)$ matrices. Our discussion is analogous to that of [65,1] and consists of obtaining a loop equation for the resolvent. In the next section, we will formulate a systematic method to obtain the g_s corrections to the classical solution.

The partition function for the model with one matrix Φ in the adjoint representation of the Lie algebra of $G = SO(M)$ or $Sp(M)$ is

$$Z = 1 \mathrm{Vol}\, G \int d\Phi \, \exp\left(-1 g_s \mathrm{Tr}\, W(\Phi)\right). \tag{4.1}$$

In Appendix A.1, we collect results that are useful for SO/Sp groups, but here we shall discuss only the $SO(2M)$ group in detail.

In the eigenvalue basis, the integral over an $SO(2M)$ matrix is given by

$$Z \sim \int \prod_{i=1}^{M} d\lambda_i \prod_{i<j} (\lambda_i^2 - \lambda_j^2)^2 \, e^{-\frac{2}{g_s}\sum_i W(\lambda_i)}. \tag{4.2}$$

The effective action for the gas of eigenvalues is given by

$$S(\lambda) = -\sum_{i<j} \ln(\lambda_i^2 - \lambda_j^2)^2 + 2 g_s \sum_i W(\lambda_i). \tag{4.3}$$

Note that W is now a polynomial of order $2n$ with only even powers. This is because the trace of an antisymmetric matrix vanishes. In principle $W(\Phi)$ could also contain the Pfaffian, but we will omit this case.

This action gives rise to the classical equations of motion

$$\sum_{j\neq i} 2\lambda_i \lambda_i^2 - \lambda_j^2 - 1 g_s W'(\lambda_i) = 0. \tag{4.4}$$

It is useful to define the resolvent

$$\omega_0(x) = g_s \mathrm{Tr}\ 1 x - \Phi = g_s \sum_i 2x x^2 - \lambda_i^2, \tag{4.5}$$

then, by multiplying the equations of motion by $\frac{2\lambda_i}{x^2-\lambda_i^2}$ and summing over i, we obtain an equation for $\omega_0(x)$ exactly as for $SU(M)$:

$$\omega_0(x)^2 - g_s \left(\frac{\omega_0(x)}{x} - \omega_0'(x) \right) + f(x) - 2\omega_0(x) W'(x) = 0, \tag{4.6}$$

where

$$f(x) = g_s \sum_i \frac{2\lambda_i W'(\lambda_i) - 2x W'(x)}{\lambda_i^2 - x^2} \tag{4.7}$$

is a polynomial of order $2n-2$ with only even powers, *i.e.*, it has n coefficients.

In the small g_s limit, (4.6) reduces to

$$\omega_0(x)^2 + f(x) - 2\omega_0(x)W'(x) = 0, \tag{4.8}$$

or

$$y^2 - W'(x)^2 + f(x) = 0, \tag{4.9}$$

where

$$y(x) = \omega_0(x) - W'(x). \tag{4.10}$$

The force equation is then

$$2y(\lambda) = -g_s \partial S \partial \lambda, \tag{4.11}$$

where the factor of 2 comes from the fact that the force is acting on an eigenvalue and its image. This is the same equation as for $SU(2M)$, with the only difference being that the polynomials W and f have only even powers. This matches the expected result from the orientifold procedure in string theory.

The equation for the resolvent can be solved using (4.9), yielding a formal solution [65]

$$\omega_0(x) = W'(x) - \sqrt{W'(x)^2 - f(x)}. \tag{4.12}$$

The resolvent is thus expressed in terms of the n unknown coefficients that appear in the polynomial $f(x)$ defined in (4.7). From the form of the solution, it is clear that the resolvent has branch cuts among which the eigenvalues of the matrix are distributed. In the large M limit, we thus get a distribution of eigenvalues, with the eigenvalue density given by $\rho(\lambda)$

$$\omega_0(x) = 2\int_0^\infty x\rho(\lambda)d\lambda x^2 - \lambda^2 = \int_0^\infty \rho(\lambda)d\lambda\,(1x - \lambda + 1x + \lambda) = \int_{-\infty}^\infty \rho(\lambda)d\lambda x - \lambda, \tag{4.13}$$

which implies that

$$\rho(\lambda) = 12\pi i(\omega_0(\lambda + i0) - \omega_0(\lambda - i0)) = 12\pi i(y(\lambda + i0) - y(\lambda - i0)). \tag{4.14}$$

The filling fractions are then given by

$$
\begin{aligned}
S_0 &= 14\pi i \int_{A_0} y(x)dx, \\
S_i &= 12\pi i \int_{A_i} y(x)dx \quad , i > 0
\end{aligned}
\tag{4.15}
$$

Note that we only integrate around half of the cycle A_0 because of the orientifold projection. At the classical level, one can see from (4.11) that $y(x)$ is the force acting on an eigenvalue. Now, the variation of the free energy $\mathcal{F}_0$ of the matrix model caused by a changing the number of eigenvalues on the i^{th} cut is then the line integral of the force over the non compact B_i cycle of the Riemann surface (4.9)

$$
\partial\mathcal{F}_0\partial S_i = \int_{B_i} y(x)dx. \tag{4.16}
$$

This is the differential equation that determines $\mathcal{F}_0$, *i.e.*, the leading contribution to the free energy.

For $SO(2M+1)$ and $Sp(M)$, one can easily see that $\mathcal{F}_0$ and the Riemann surface are the same as in the case of $SO(2M)$. In the next section we will determine the leading contribution from unoriented diagrams to the free energy, which is a subleading term in the g_s expansion of the free energy.

5 g_s Corrections and Loop Equations

The partition function of the SO/Sp matrix model is

$$
Z = e^{\frac{1}{g_s^2}\mathcal{F}} = \int d\Phi e^{-\frac{1}{g_s}\mathrm{Tr}\ W(\Phi)}, \tag{5.1}
$$

where the overall coupling constant g_s can be thought of as the string coupling and the action is

$$
W(\Phi) = \sum_{j=1}^{\infty} \frac{g_j}{2j}\Phi^{2j}. \tag{5.2}
$$

Dijkgraaf and Vafa [1–3] conjectured that the exact superpotential of the gauge theory with the tree level superpotential $W(\Phi)$ is given by the perturbative expansion of the matrix integral (5.1) around one classical vacuum (saddle point). Such a classical vacuum is given by a distribution of the eigenvalues of Φ over the critical points $\{x_i\}$

of the superpotential $W(x)$. We denote the number of eigenvalues at the critical point x_i by M_i and define the parameters

$$S_0 = g_s \frac{M_0}{2}, \quad S_i = g_s M_i. \tag{5.3}$$

The perturbative expansion around such a classical vacuum can be visualized in terms of fat graphs, where edges of a ribbon correspond to Chan-Paton factors. For each Chan-Paton factor we have to choose a critical point x_i, on which it sits and a loop of such a Chan-Paton factor gives a contribution of $M_i = S_i/g_s$. From the overall normalization of the action, it is clear that each vertex of the diagram contributes a factor of $1/g_s$ and each propagator contributes g_s. Thus the overall power of g_s counts the Euler character of the fat graph. The superpotential is then given only by the contributions from planar and $\mathbb{RP}^2$ diagrams. These diagrams have a very simple g_s dependence, but the S_i dependence can actually be quite complicated, as we will see.

If one did the full matrix integral, there would be a sum over all saddle points and the S_i dependence would be lost. However, since we are interested in only in a perturbative expansion around a classical vacuum, the S_i dependence is nontrivial and will describe how the effective superpotential depends on the gaugino condensates.

In a recent paper [66], it has been shown that the in a vacuum where the gauge symmetry is broken to a subgroup (say, a product of $U(N_i)$ factors), the off diagonal components of the matrix Φ do not correspond to propagating degrees of freedom, and that these should be properly interpreted as the Faddeev-Popov ghosts that are necessarily included because of the gauge fixing involved in doing the matrix model. Thus, for computing Feynman diagrams in the matrix model, we have to include terms in the Lagrangian that belong to the ghost sector as well. But the loop equations, as we shall see, correspond to Ward identities in the matrix model. They arise because of the invariance of the matrix integral under an arbitrary reparametrization of Φ that respects the SO/Sp symmetry of the Lagrangian. If we take into account the variation of the measure as well, then this symmetry leads to the loop equations. Thus, we do not expect the ghosts to be relevant for the discussion in this section.

In the SO/Sp case we expand the matrix model partition function in a systematic expansion in g_s. The coefficients of the terms in the expansion are the contributions coming from the Feynman graphs that can be drawn on a surface of Euler character $\chi = 2 - 2g - c$ where g denotes the genus, and c denotes the number of cross-caps. We mentioned earlier that each loop in a Feynman diagram contributes a factor M.

In order to see this, consider the propagator for the $SO(M)$ matrix model. It has a group theoretic factor

$$\langle \Phi_{ij} \Phi_{kl} \rangle \sim \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{5.4}$$

Thus, each loop in a Feynman diagram contributes a factor of M_i (the number of Chan-Paton factors on the ith critical point).

5.1 The Resolvent

We shall now introduce the general technique of loop equations, which is an iterative procedure to calculate the higher order (in g_s) corrections to the partition function. Central to this procedure is the loop operator defined as

$$\frac{d}{dV}(x) = -\sum_{j=1}^{\infty} \frac{2j}{x^{2j+1}} \frac{\partial}{\partial g_j}. \tag{5.5}$$

The resolvent, which is the generating functional for the single trace correlation functions of the matrix model is defined as

$$\omega(x) = g_s \left\langle \mathrm{Tr}\, \frac{1}{x - \Phi} \right\rangle = g_s \sum_{k=0}^{\infty} \frac{\langle \mathrm{Tr}\ \Phi^{2k} \rangle}{x^{2k+1}} \tag{5.6}$$

Using the identity

$$-(2k)\frac{d}{dg_k}\mathcal{F} = g_s \langle \mathrm{Tr}\ \Phi^{2k} \rangle, \tag{5.7}$$

we can express the resolvent as

$$\omega(x) = \frac{d}{dV}(x)\mathcal{F} + \frac{S}{x}, \tag{5.8}$$

where we used $S = \sum S_i = g_s M$. We are using the variables g_s and S, since we are working in the small g_s limit with S fixed. As mentioned before, the free energy has an expansion in g_s of the form

$$\mathcal{F} = \sum_{g,c} g_s^{2g+c} \mathcal{F}_{g,c} \tag{5.9}$$

We will be interested in calculating the first two terms in this expansion, which are the contributions from diagrams with the topology of S^2 and $\mathbb{RP}^2$. The resolvent has a similar expansion

$$\omega(x) = \sum_{g,c} g_s^{2g+c} \omega_{g,c}(x), \tag{5.10}$$

and the S_i

$$S_i = \sum_{g,c} g_s^{2g+c} S_i^{(g,c)} \tag{5.11}$$

The asymptotic behavior of the $\omega_{g,c}$ functions are clear from the definition of $\omega(x)$.

$$\lim_{x\to\infty} \omega_{g,c}(x) \sim \frac{S^{(g,c)}}{x}, \tag{5.12}$$

Using this fact, and the existence of the genus expansion, we can write

$$\omega_{g,c}(x) = \frac{d}{dV}(x)\mathcal{F}_{g,c} + \frac{S^{(g,c)}}{x} \tag{5.13}$$

These equations determine the dependence of $\mathcal{F}_{g,c}$ on the coupling constants. There is still an overall constant, which is undetermined and actually unphysical as well. In the next section, we will derive the loop equation, which will provide us with recursion relations to calculate $\omega_{g,c}$ as functions of the coupling constants g_j. (For the rest of the discussion, we denote $\omega_{0,0}$ by ω_0 and $\omega_{0,1}$ by ω_1.)

5.2 The Loop Equation

The loop equation can be derived by doing a reparametrization of the matrices Φ in the matrix integral and observing that the integral is invariant under this reparametrization. Let us reparametrize Φ by

$$\Phi = \Phi' - \left(\frac{\epsilon}{x-\Phi'}\right)_{\rm odd} = \Phi' - \epsilon\sum_{k=0}^{\infty}\frac{\Phi'^{2k+1}}{x^{2k+2}} \tag{5.14}$$

$$d\Phi = d\Phi' - \epsilon\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\frac{\Phi'^{l}d\Phi'\Phi'^{2k-l}}{x^{2k+2}} = d\Phi' - \epsilon\left(\frac{1}{x-\Phi'}d\Phi'\frac{1}{x-\Phi'}\right)_{\rm even} \tag{5.15}$$

where we only take the odd/even powers of Φ' in order to preserve the SO/Sp Lie algebra. The Jacobian for this reparametrization is then

$$J(\Phi') = 1 - \epsilon\left(\mathrm{Tr}\,\frac{1}{x-\Phi'}\right)^2 \tag{5.16}$$

The action transforms as

$$\mathrm{Tr}\,W(\Phi) = \mathrm{Tr}\,W\left(\Phi' - \left(\frac{\epsilon}{x-\Phi'}\right)_{\rm odd}\right) = \mathrm{Tr}\,W(\Phi') - \epsilon\mathrm{Tr}\,\frac{W'(\Phi')}{x-\Phi'} \tag{5.17}$$

Inserting this into the matrix integral, we get

$$\int d\Phi' \left(\text{Tr} \, \frac{1}{x - \Phi'} \right)^2 e^{-\frac{1}{g_s} \text{Tr} \ W(\Phi')} = \frac{1}{g_s} \int d\Phi' \text{Tr} \, \frac{W'(\Phi')}{x - \Phi'} e^{-\frac{1}{g_s} \text{Tr} \ W(\Phi')} \tag{5.18}$$

We can now make use of the identity

$$\frac{d}{dV}(x)\omega(x) = \left\langle \left(\text{Tr} \, \frac{1}{x - \Phi} \right)^2 \right\rangle - \left\langle \text{Tr} \, \frac{1}{x - \Phi} \right\rangle^2 \tag{5.19}$$

to get the loop equation

$$g_s \left\langle \text{Tr} \, \frac{W'(\Phi)}{x - \Phi} \right\rangle = \omega(x)^2 + g_s^2 \frac{d}{dV}(x)\omega(x). \tag{5.20}$$

We can rewrite the loop equation using

$$g_s \left\langle \text{Tr} \, \frac{W'(\Phi)}{x - \Phi} \right\rangle = g_s \left\langle \sum_i \frac{W'(\lambda_i)}{x - \lambda_i} \right\rangle = \oint_C \frac{dx'}{2\pi i} \frac{W'(x')}{x - x'} \omega(x'), \tag{5.21}$$

where C is a contour that encloses all the eigenvalues of Φ but not x. In the small g_s (large M) limit of the matrix model, we get a continuous eigenvalue distribution for Φ, and all the eigenvalues are distributed over cuts on the real axis of the x-plane. The loop equation now reads

$$\oint_C \frac{dx'}{2\pi i} \frac{W'(x')}{x - x'} \omega(x') = \omega(x)^2 + g_s^2 \frac{d}{dV}(x)\omega(x). \tag{5.22}$$

We can now insert the g_s expansions for the resolvent, and iteratively solve for the $\omega_{g,c}$. The zeroth and first order equations are

$$\oint_C \frac{dx'}{2\pi i} \frac{W'(x')}{x - x'} \omega_0(x') \ = \ \omega_0(x)^2 \tag{5.23}$$

$$\oint_C \frac{dx'}{2\pi i} \frac{W'(x')}{x - x'} \omega_1(x') \ = \ 2\omega_0(x)\omega_1(x) \tag{5.24}$$

The resolvent that solves the loop equations has to satisfy (5.12) which imposes constraints on the end points of the cuts in the x-plane. Note that this derivation of the loop equation is valid in the saddle point approximation that we are using.

Equation (5.24) is solved by a derivative of ω_0 with respect to any parameter, which specifies the vacuum, *i.e.*, is independent of the coupling constants g_j. In our case there are only the parameters S_i, which specify the classical vacuum around which the matrix integral is expanded. We will elaborate on this observation in section 5.3.2 for the case of a softly broken $\mathcal{N} = 2$ theory and we will use this result for the $\mathcal{N} = 1^*$ theory.

5.3 Solution to the Loop Equations

Let us now solve the the loop equations (5.23) first for ω_0 and then for ω_1 in the case of a polynomial potential

$$W(\Phi) = \sum_{j=1}^{n} \frac{g_j}{2j} \Phi^{2j}. \tag{5.25}$$

In this section, we closely follow the discussion in [30, 31].

5.3.1 Planar Contributions

In equation (5.23), we deform the integration contour C to encircle infinity, and rewrite it as

$$\omega_0(x)^2 = W'(x)\omega_0(x) + \oint_{C_\infty} \frac{dx'}{2\pi i} \frac{W'(x')\omega_0(x')}{x - x'} \tag{5.26}$$

Assuming that $\omega_0(x)$ has k cuts in the complex x-plane, we make the following ansatz

$$\omega_0(x) = \frac{1}{2}\left(W'(x) - M(x)\sqrt{\prod_{i=1}^{2k}(x - x_i)} \right) \tag{5.27}$$

where $M(x)$ is an undetermined analytic function at the moment. Here, the end points of the cuts denoted by the x_i are unknown and have to be determined. From the discussion in [31], it is clear that if we have the maximum number of cuts $k = 2n-1$ allowed, the function $M(x)$ is a constant. The loop equation determines M in this case to be the coupling constant g_n. Also, in the SO/Sp case the eigenvalues come in pairs and the total number of "independent" cuts is n. There is one cut $[-x_0, x_0]$ centered around zero, and the other cuts come in pairs $[x_{2i-1}, x_{2i}]$ and $[-x_{2i}, -x_{2i-1}]$. We shall follow these notations in what follows.

We now demand that the resolvent $\omega_0(x)$ have the $1/x$ fall off at infinity, and thus get n constraints

$$\delta_{k,n} = \frac{1}{2}\oint_C \frac{dx'}{2\pi i} \frac{x'^{2k-1} W'(x')}{\sqrt{\prod_{i=0}^{2(n-1)}(x'^2 - x_i^2)}}, \quad k = 1, 2, \cdots, n. \tag{5.28}$$

The most general solution to these n constraints (5.28) is given by

$$g_n^2 \prod_{i=0}^{2(n-1)} (x^2 - x_i^2) = W'(x)^2 - f(x), \tag{5.29}$$

where $f(x)$ is the most general even polynomial of order $2n-2$.

$$f(x) = \sum_{l=0}^{n-1} b_l x^{2l} \tag{5.30}$$

Note that we have now recovered the solution to the classical loop equation that we obtained in an earlier section. We now repeat the procedure outlined there and define the Riemann surface Σ given by

$$y^2 = W'(x)^2 - f(x). \tag{5.31}$$

The filling fractions S_i then become period integrals of the meromorphic 1-form $y\,dx$ over the 1-cycle A_i of Σ, that encircles the i^{th} branch cut

$$S_i = \oint_{A_i} \frac{y\,dx}{2\pi i}. \tag{5.32}$$

We can then argue that the change in the free energy due to bringing an eigenvalue from infinity to the i'th cut is

$$\frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{B_i} y\,dx. \tag{5.33}$$

Note here that the B cycles are non compact, and for (5.33) to make sense, we have to introduce an ultraviolet cut-off Λ_0 in the integral which has been identified with the bare coupling of the gauge theory [44]. We comment here that there are only semi-classical arguments for equation (5.33), and we have been unable to rigorously prove this as a consequence of the loop equations and (5.8).

5.3.2 $\mathbb{RP}^2$ *Contributions*

Once we have the form of the solution for $\omega_0(x)$, we can substitute it in the loop equation, which is now linear in $\omega_1(x)$

$$\oint_C \frac{dx'}{2\pi i} \frac{W'(x')\omega_1(x')}{x - x'} = 2\omega_0(x)\omega_1(x). \tag{5.34}$$

The general solution ω_1 of the first order loop equation is

$$\omega_1(x) = \frac{P(x)}{\sqrt{W'(x)^2 + f(x)}}, \tag{5.35}$$

with $P(x) = \sum_l c_l x^{2l}$ any polynomial of order $2n-2$, which has n undetermined constants. The coefficients c_l in the polynomial can still be arbitrary functions of the coupling constants g_j and the S_i. These coefficients can be determined from the requirement, that there has to be a function $\mathcal{F}_1(g_j, S_i)$, such that (5.13) is satisfied.

We can get a natural ansatz for $\mathcal{F}_1$ and ω_1 from the observation that a derivative with respect to S_0 of ω_0 solves the first order loop equation.

$$\mathcal{F}_1 = q\frac{\partial \mathcal{F}_0}{\partial S_0^{(0)}}, \tag{5.36}$$

where q is an arbitrary constant. Inserting this into (5.13), we get

$$\begin{aligned} \frac{d}{dV}(x)\mathcal{F}_1 =& -q\sum_j \frac{(2j)}{x^{2j+1}}\frac{\partial}{\partial g_j}\frac{\partial \mathcal{F}_0}{\partial S_0^{(0)}} \\ =& q\frac{\partial}{\partial S_0}\left(\omega_0(x) + \frac{S^{(0)}}{x}\right) \\ =& q\frac{\partial \omega_0}{\partial S_0^{(0)}} + \frac{q}{x}. \end{aligned} \tag{5.37}$$

From this we see, that we have to set $S_0^{(1)} = q$. Similarly we could also make the ansatz with $S_i^{(0)}$ replacing the $S_0^{(0)}$ in (5.36). However, corrections $S_i^{(1)}$ for $i > 0$ should not be generated since these cuts represent $U(N_i)$ gauge physics. In order to exclude this possibility, we need to go beyond the loop equation and calculate the first perturbative contributions to the free energy that come from diagrams with the topology of S^2 and $\mathbb{RP}^2$. This will fix $q = \pm\frac{g_s}{4}$.

5.4 Counting Feynman diagrams with S^2 and $\mathbb{RP}^2$ topology

In order to fix the undetermined coefficient q in the relation

$$\mathcal{F}_1 = \pm q\frac{\partial \mathcal{F}_0}{\partial S_0} \tag{5.38}$$

we need to enumerate "ribbon" graphs in the 't Hooft (genus) expansion of the matrix model. Recall that the genus expansion is ordered by diagram topology, with diagrams of genus g and c cross-caps contributing at order $g_s^{-\chi} = g_s^{-2+2g+c}$. The coefficient q is related to the relative contribution of the planar (genus 0) diagrams which dominate at large M with the leading $\frac{1}{M}$ correction, coming from diagrams with topology $\mathbb{RP}^2$.

It is known that $SO(2M)$ and $Sp(M)$ matrix models are related by analytic continuation $M \mapsto -M$ (for the analogous gauge theory results see [67,68,33]). Therefore, at even orders in the genus expansion, the contribution to the matrix model free energy is the same for both theories, while at odd orders the $Sp(M)$ diagrams contribute to the free energy with an additional minus sign relative to $SO(2M)$. This fact determines the sign in (5.38).

Recall that

$$\chi = v - p + l \tag{5.39}$$

where v is the number of vertices in the ribbon graph, p is the number of propagators and l the number of boundary loops. The Feynman rules are summarized in appendix C Let us evaluate the first-order quartic diagrams in fig. 1. The planar diagram has the value

$$2 \times \frac{1}{1!} \frac{g_2}{4g_s} \left(\frac{g_s}{2m}\right)^2 M^3 \tag{5.40}$$

whereas the $\mathbb{RP}^2$ diagram with one twisted propagator contributes

$$-4 \times \frac{1}{1!} \frac{g_2}{4g_s} \left(\frac{g_s}{2m}\right)^2 M^2 \tag{5.41}$$

and the the $\mathbb{RP}^2$ diagram with two twisted propagators contributes

$$1 \times \frac{1}{1!} \frac{g_2}{4g_s} \left(\frac{g_s}{2m}\right)^2 M^2. \tag{5.42}$$

This shows that

$$\mathcal{F}_1 = -\frac{1}{2}\frac{\partial \mathcal{F}_0}{\partial M} = -g_s \frac{1}{4}\frac{\partial \mathcal{F}_0}{\partial S_0} \tag{5.43}$$

at the first order.

We have calculated the Feynman diagrams for several higher orders and higher vertices and confirmed this relationship in those cases[4].

We can now normalize the result in the loop equations and extend it to a single cut model with an arbitrary polynomial potential. In the case that there is no quartic interaction, we can analytically continue the result from a potential of the same order with a quartic interaction. Since this is true for polynomial interactions of all orders, it actually has to be true even for nonpolynomial interactions. We will use this to solve the SO/Sp $\mathcal{N} = 1^*$ theories. It would be nice to have a purely combinatorial proof of (5.43), that would not rely on the loop equation.

[4]This relationship was apparently not known to mathematicians.

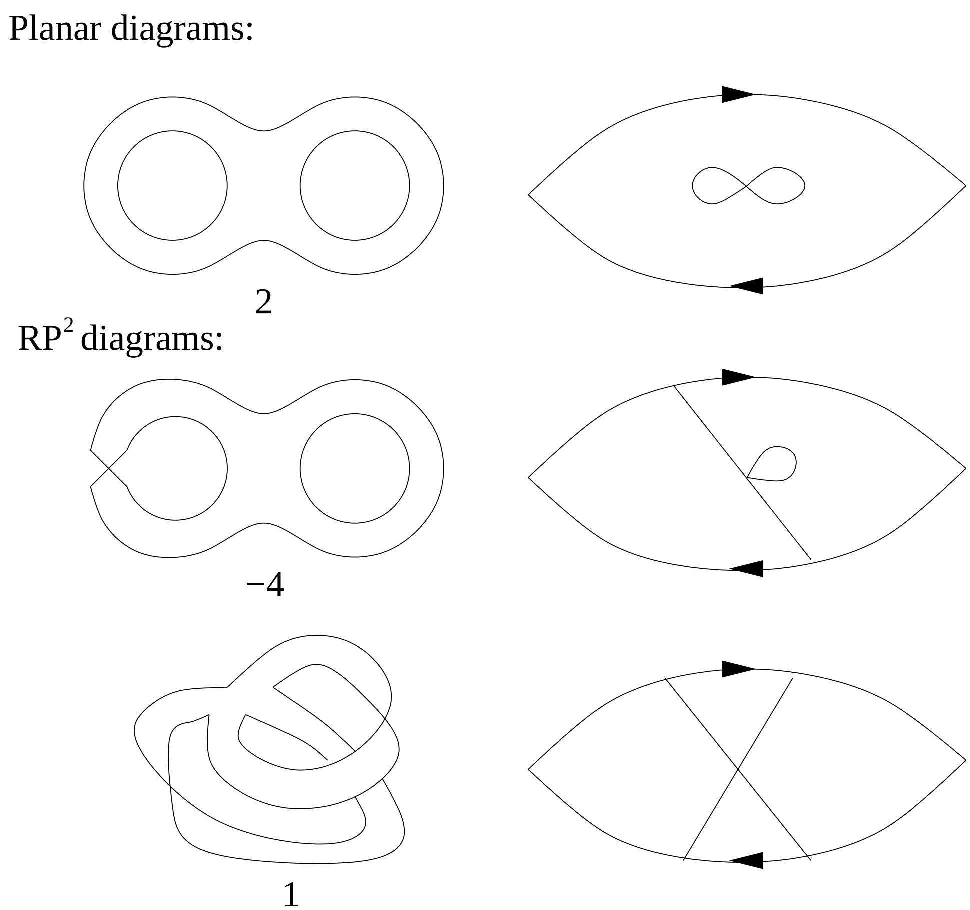

Figure 1: S^2 and $\mathbb{RP}^2$ diagrams with one quartic vertex, written in terms of twisted and untwisted propagators and as diagrams on $\mathbb{RP}^2$ to show their planarity. Propagators that pass through the cross-cap become twisted.

In order to describe a multi-cut matrix model, we need to use ghosts to expand around the classical vacuum. In this prescription, one can think of the matrix model as several matrix models, which are coupled by bifundamental ghosts. Only one of those matrix models is actually an $SO(M_0)/Sp(M_0/2)$ matrix model with twisted propagators. The other matrix models are $U(M_i)$ matrix models. The ghosts do not have twisted propagators either, so the leading contribution from the $SO(M_0)/Sp(M_0/2)$ matrix model is again the same as for a single cut model. The loop equations still hold for the multi-cut model and again we can extend the result to all orders.

6 Computation of Effective Superpotentials

In this section we combine the results of the previous sections to compute the effective superpotential of the dual gauge theories. We will find that it is necessary to refine the formula for the unoriented string contribution to the effective superpotential of [3].

6.1 Non-Perturbative Sector

As discussed in [69,3], there is a non-perturbative contribution to the free energy which arises from the Gaussian integral:

$$\mathcal{F}^{np} = \frac{1}{2}\dim G \, \log \frac{2\pi g_s}{m} - \log\,\mathrm{vol}(G). \tag{6.1}$$

In appendix B, following [69], we have included the large M expansion of the logarithm of the volume of the SO/Sp groups. We find that, for $SO(M)$ when M is even,

$$\begin{aligned}
\mathcal{F}^{np} =& \frac{1}{g_s^2}\mathcal{F}_0^{np} + \frac{1}{g_s}\mathcal{F}_1^{np} + \cdots \\
=& \frac{1}{g_s^2}\left[S^2 \log \frac{2\pi S}{m} - S^2\left(\frac{3}{2} + \log \pi\right)\right] \\
&+ \frac{1}{g_s}\left[-\frac{S}{2}\log \frac{2\pi S}{m} + \frac{S}{2}(1 + \log \pi - \log 4)\right] + \cdots ,
\end{aligned} \tag{6.2}$$

with a similar expression for M odd or $G = Sp(M)$. We see that

$$\mathcal{F}_1^{np} = \mp \frac{1}{4}\frac{\partial \mathcal{F}_0^{np}}{\partial S} \pm \frac{1}{2}\log 2, \tag{6.3}$$

where the first $-/+$ sign is for SO/Sp respectively. This is almost the same relationship as we found for the perturbative contributions (5.43), but it is spoiled by the $\log 2$ term.

We have traced this term through the volume computation outlined in [69] and found that it could be removed by a different choice for the measure on the maximal torus of the Lie group.

It is the non-perturbative sector, specifically the coefficient of the $S^2 \log S$ term, that determines the number of gauge theory vacua, which is a main consistency test of the translation between matrix model quantities and the effective superpotential of the gauge theory. The number of vacua of a supersymmetric gauge theory is equal to the dual Coxeter number h of the gauge group [70, 71]. Therefore the total superpotential should lead to the conclusion that S^h is single-valued.

Open string physics tells us that the sphere contribution to the effective superpotential should be proportional to Q_{D5}, the total charge of D5-branes, while the $\mathbb{RP}^2$ contribution should be proportional to Q_{O5}, the total charge of O5-planes. We can express this by refining the suggestion of [3]:

$$W_{\text{eff}} = Q_{D5}\, \partial\mathcal{F}_0 \partial S + Q_{O5}\, \mathcal{G}_0 - 2\pi i\, \tau\, S, \tag{6.4}$$

We assume that $\mathcal{G}_0$ is proportional to the total $\mathbb{RP}^2$ free energy,

$$\mathcal{G}_0 = a\, (\mathcal{F}_1^{np} + \mathcal{F}_1^{p})\,. \tag{6.5}$$

Proceeding with this result, we find that

$$W_{\text{eff}} = \left(\frac{N}{2} \pm \frac{a}{4}\right) S \log S - \frac{1}{2} \tau\, S + \cdots\,, \tag{6.6}$$

where the $+/-$ is for SO/Sp respectively. Consistency with both the closed string result (3.10) and the gauge theory[5] requires that we must have $a = \mp 4$. Very recently [35] produced this factor $|a| = 4$ from a perturbative argument along the lines of [28]. It was found to be related to the measure on the moduli space of Schwinger parameters, a quantity which is intrinsic to the gauge theory. Presumably, there is a similar explanation of this correction within the holomorphic Chern-Simons theory.

6.2 The $\mathcal{N} = 1^*$ Theories

Following [3], we can also consider the $\mathcal{N} = 1^*$ theories. Table 1 contains the results that are needed to write the partition function in the eigenvalue basis. In the large M

[5] Note that, after including $a = \mp 4$, the effective superpotential naively suggests that for gauge group $Sp(N/2)$, S^{N+2} is single-valued, whereas $h = N/2 + 1$. The resolution to this puzzle was explained in [59]. Namely the D1-string wrapped on $\mathbb{P}^1$ has instanton number *two* in $Sp(N/2)$. Properly accounting for this reproduces the $\mathbb{Z}_{2h}$ chiral symmetry of the dual gauge theory.

limit, the discussion will entirely parallel that of [3]. Inclusion of $\mathbb{RP}^2$ contributions to the superpotential for $SO(2N)$ gauge group yields

$$W_{\text{eff}} = \frac{2N-2}{2}\,\Pi_B(\tau) - \frac{1}{2}\tau_0\,\Pi_A(\tau). \tag{6.7}$$

This has extrema at

$$\tau = \frac{\tau_0 + k}{2N-2},\ \ k = 0, \cdots 2N-1, \tag{6.8}$$

at which points the superpotential takes the critical values

$$W_{(k)} \sim E_2((\tau_0 + k)/(2N-2)), \tag{6.9}$$

in complete agreement with (2.16).

7 Discussion

In this paper we have outlined a general scheme for computing the subleading contributions to the gauge theory effective superpotential from $\mathbb{RP}^2$ diagrams in the dual matrix model of Dijkgraaf and Vafa. The methods involve an application of the higher-genus loop equations to determine the $\mathbb{RP}^2$ correction to the resolvent, which allows us to compute the $\mathbb{RP}^2$ contribution to the free energy of the matrix model. We then established a refinement of Dijkgraaf and Vafa's relationship between matrix model quantities and the gauge theory effective superpotential which was necessary to obtain consistent field theoretic results. The computation of [35] provides a gauge theoretic explanation of our prescription.

There are many future directions that can be pursued. First, while we were interested in an order g_s (equivalently $1/M$) correction from unoriented diagrams, in the matrix model duals to gauge theory with matter in general representations, such as quarks in the fundamental, there will be $1/M$ corrections[6] arising from worldsheets with a single boundary. Our application of the loop equations should apply to this case and it would be interesting to use this to make contact with the results of [13–16, 12].

Similarly, it would be of interest to examine higher-order corrections in the genus expansion, which have an interpretation as gravitational corrections [1, 3, 21, 22]. It

[6]For example, for $N_f < N_c$, these correspond to terms of order N_f/N_c in the gauge theory effective superpotential.

would be interesting to make contact between the loop equations and the Kodaira-Spencer equations of [72], which also relate higher genus results to those at lower genus. It seems reasonable that there are higher genus forms of our relation between oriented and unoriented contributions at a given genus like

$$\mathcal{F}_{g,1} \propto \frac{\partial \mathcal{F}_{g,0}}{\partial S_0}. \tag{7.1}$$

It would also be useful to obtain a deeper understanding of relations like (7.1) from the diagrammatic combinatorics. For example a diagrammatic proof of our conjecture seems to be possible.

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A Matrix Integral Measures and Determinants

In this section we collect some results on the group measure and adjoint action which are needed to do computations in the matrix models.

A.1 The Group Measure for General Matrices

We wish to compute the Jacobian for the transformation from certain matrices Φ to their eigenvalues. This can be derived by a group-theoretic argument. In terms of the Cartan generators H_i and ladder operators E_α, for the algebra of the group G, satisfying

$$[H^i, E^\alpha] = \alpha^i E^\alpha, \tag{A.1}$$

we can diagonalize a matrix Φ

$$\begin{aligned}\Phi &= U^\dagger \Lambda U,\\ \Lambda &= \sum_i \lambda_i H^i.\end{aligned} \tag{A.2}$$

We will define parameters t_α so that

$$dU = \left[\sum_\alpha dt_\alpha E_\alpha\right] U, \quad t^*_\alpha = -t_{-\alpha}. \tag{A.3}$$

The infinitesimal variation of Φ can then be written as

$$\begin{aligned} d\Phi =& U^\dagger \left[d\Lambda + \sum_\alpha dt_\alpha \left[\Lambda, E^\alpha\right]\right] U \\ =& U^\dagger \left[d\Lambda + \sum_\alpha dt_\alpha \left(\sum_i \lambda_i \alpha^i\right) E^\alpha\right] U. \end{aligned} \tag{A.4}$$

We now calculate the metric on the Lie algebra

$$\mathrm{Tr}\, d\Phi\, d\Phi^\dagger = \sum_i d\lambda_i^2 + \sum_{\alpha,\beta} dt_\alpha\, dt_\beta \left(\sum_i \lambda_i \alpha_i\right)\left(\sum_j \lambda_j \beta_j\right) \mathrm{Tr}\, E_\alpha E_\beta. \tag{A.5}$$

Using the identity

$$\mathrm{Tr}_r E^\alpha E^\beta = C(r)\delta_{\alpha+\beta,0} \tag{A.6}$$

where $C(r)$ is a representation dependent constant, we can simplify the second term in equation (A.5) to

$$C(r)\sum_\alpha \left|\sum_i \alpha^i \lambda_i\right|^2 |dt_\alpha|^2 \tag{A.7}$$

Up to numerical factors, the Jacobian is

$$\Delta(\Lambda) = \prod_{\alpha>0} \left|\sum_i \alpha^i \lambda_i\right|^2. \tag{A.8}$$

A.2 The Induced Measure of the $\mathcal{N} = 1^*$ Theory

In this section, we calculate $\det\left(\mathrm{adj}_\Phi + im\right)$ whose modulus squared appears in the calculation of the induced measure of the $\mathcal{N} = 1^*$ theory. In order to calculate the determinant, we go to a diagonal basis in which Φ is an element of the Cartan subalgebra (A.2). Then we solve the eigenvalue equation

$$[\Phi, A] + iA = aA, \tag{A.9}$$

where A is a completely general matrix in the Lie algebra

$$A = \sum_i h_i H^i + \sum_\alpha t_\alpha E^\alpha. \tag{A.10}$$

We can compute

$$[\Phi, A] + iA = i \sum_i h_i H^i + \sum_\alpha \left(\sum_i \alpha^i \lambda_i + i \right) t_\alpha E^\alpha, \tag{A.11}$$

so the eigenvectors and eigenvalues are

$$\begin{aligned} A &= H^i, \quad a = i, \\ A &= E_\alpha, \quad a = \big(\sum_i \alpha^i \lambda_i + i\big). \end{aligned} \tag{A.12}$$

Up to numerical factors, the determinant is then

$$\det\left(\mathrm{adj}_\Phi + im\right) \sim \prod_{\alpha>0} \left(\sum_i \alpha^i \lambda_i + im \right). \tag{A.13}$$

We list the expressions for the roots and the corresponding determinants for the different classical groups in Table 1.

B Asymptotic expansion of the gauge group volumes

We now compute the asymptotic expansion of the volume of the gauge groups, which normalizes the partition function of the matrix model and provides the nonperturbative contribution to the free energy. The volumes are given by [69]:

$$\begin{aligned} \mathrm{vol}(SO(2N+1)) &= 2^{N+1}(2\pi)^{N^2+N-14}(2N-1)!(2N-3)!\ldots 3!1!, \\ \mathrm{vol}(SO(2N)) &= \sqrt{2}(2\pi)^{N^2}(2N-3)!(2N-5)!\ldots 3!1!(N-1)!, \\ \mathrm{vol}(Sp(2N)) &= 2^{-N}(2\pi)^{N^2+N}(2N-1)!(2N-3)!\ldots 3!1!. \end{aligned} \tag{B.1}$$

G Roots	$J(\Lambda)$ $\det(\mathrm{adj}_\Lambda + i)$
A_{N-1} $e_i - e_j\ (i \neq j)$	$\prod_{i<j}(\lambda_i - \lambda_j)^2$ $\prod_{i<j}(\lambda_i - \lambda_j + i)(\lambda_i - \lambda_j - i)$
B_N $\pm e_i \pm e_j\ (i \neq j),\ \pm e_i$	$\prod_{i<j}(\lambda_i^2 - \lambda_j^2)^2 \prod_i \lambda_i^2$ $\prod_{i<j}((\lambda_i - \lambda_j)^2 + 1)((\lambda_i + \lambda_j)^2 + 1) \prod_i(\lambda_i^2 + 1)$
C_N $\frac{1}{\sqrt{2}}(\pm e_i \pm e_j)\ (i \neq j),\ \pm\sqrt{2}e_i$	$\prod_{i<j}(\lambda_i^2 - \lambda_j^2)^2 \prod_i \lambda_i^2$ $\prod_{i<j}((\lambda_i - \lambda_j)^2 + 2)((\lambda_i + \lambda_j)^2 + 2) \prod_i(2\lambda_i^2 + 1)$
D_N $\pm e_i \pm e_j\ (i \neq j)$	$\prod_{i<j}(\lambda_i^2 - \lambda_j^2)^2$ $\prod_{i<j}((\lambda_i - \lambda_j)^2 + 1)((\lambda_i + \lambda_j)^2 + 1)$

Table 1: The roots and the formulæ for the Jacobians and determinants of the adjoint actions for the classical groups.

We are interested in the large N asymptotic expansion of the logarithm of the volumes in order to compute the non-perturbative contribution to the free energy. Following [69], we introduce the Barnes function

$$G_2(z+1) = \Gamma(z) G_2(z),\ \ G_2(1) = 1. \tag{B.2}$$

Using the doubling formula for $\Gamma(z)$,

$$\Gamma(2z) = 2^{2z-1}\pi^{-12}\Gamma(z)\Gamma(z+12), \tag{B.3}$$

and (B.2), can evaluate the denominator of the volume factors

$$G_d(N) \equiv (2N-1)! \ldots 3!1! = 1(4\pi)^{N/2} 2^{N(N+1)} G_2(N+1) G_2(N+32) \tag{B.4}$$

Using the Binet integral formula

$$\log\Gamma(z) = (z-12)\log z - z + 12\ \log\ 2\pi + 2\int_0^\infty \tan(tz) e^{2\pi t} - 1 dt, \tag{B.5}$$

the asymptotic expansion of $G_2(n)$ is

$$\log G_2(N+1) = N^2 2 \log N - 112 \log N - 34N^2 + 12N \log 2\pi + O(1). \tag{B.6}$$

By expanding $\log(N-a)$ for large N, we obtain

$$\begin{aligned}\log G_d(N) =& N^2 \log N + N^2(-32 + \log 2)\\ &+ 12N \log N - 124 \log N + N2(\log 4\pi - 1) + O(1).\end{aligned} \tag{B.7}$$

Putting all of this together, we find that

$$\begin{aligned}&\log \mathrm{vol}(SO(2N+1))\\ &\qquad = -N^2 \log N + N^2(32 + \log \pi)\\ &\qquad\quad - 12N \log N + 124 \log N + N2(1 + \log 4 + \log \pi) + O(1),\\ &\log \mathrm{vol}(SO(2N))\\ &\qquad = -N^2 \log N + N^2(32 + \log \pi)\\ &\qquad\quad + 12N \log N + 124 \log N + N2(-1 + \log 4 - \log \pi) + O(1),\\ &\log \mathrm{vol}(Sp(2N))\\ &\qquad = -N^2 \log N + N^2(32 + \log \pi)\\ &\qquad\quad - 12N \log N + 124 \log N + N2(1 - \log 4 + \log \pi) + O(1).\end{aligned} \tag{B.8}$$

C Matrix model Feynman rules and enumeration of diagrams

We want to perturbatively evaluate the matrix integral

$$\int d\Phi\, e^{\frac{1}{g_s}\mathrm{Tr}\ W(\Phi)}, \tag{C.1}$$

where the potential W is given by

$$W(\Phi) = \sum_{j=1}^{\infty} \frac{g_j}{2j} \Phi^{2j} \tag{C.2}$$

and Φ is a real antisymmetric $M \times M$ matrix. We can write this as

$$\int d\Phi\ \exp\left[\frac{1}{g_s}\mathrm{Tr}\ \left(\frac{m}{2}\Phi^2 + \sum_{j=2}^{\infty} \frac{g_j}{2j}\Phi^{2j}\right)\right], \tag{C.3}$$

where $m = g_1$. Expanding the exponential leads to traces of integrals of the form

$$\begin{aligned}&\int d\Phi\, e^{\frac{1}{g_s}\mathrm{Tr}\ \frac{m}{2}\Phi^2}\, \Phi_{m_1 n_1} \cdots \Phi_{m_k n_k} =\\ &\qquad \frac{\partial}{\partial J_{m_1 n_1}} \cdots \frac{\partial}{\partial J_{m_k n_k}} \left(\int d\Phi\ \exp\left[\frac{1}{g_s}\mathrm{Tr}\ \frac{m}{2}\Phi^2 - \frac{1}{2}\mathrm{Tr}\ J\Phi\right]\right)_{J=0}.\end{aligned} \tag{C.4}$$

This integral can now be evaluated, leading to

$$\left(\sqrt{\frac{2\pi g_s}{m}}\right)^{\frac{M(M-1)}{2}} \frac{\partial}{\partial J_{m_1 n_1}} \cdots \frac{\partial}{\partial J_{m_k n_k}} \left(e^{-\frac{g_s}{8m}\mathrm{Tr}\ J^2}\right)_{J=0}. \tag{C.5}$$

Differentiating step by step gives rise to expressions like

$$\begin{aligned}
&\frac{\partial}{\partial J_{mn}}\left(\frac{g_s}{2m}J_{m_1n_1}\cdots\frac{g_s}{2m}J_{m_kn_k}e^{-\frac{g_s}{8m}\mathrm{Tr}\ J^2}\right)\\
&\qquad\qquad = \frac{g_s}{2m}(\delta_{mm_1}\delta_{nn_1}-\delta_{mn_1}\delta_{nm_1})\frac{g_s}{2m}J_{m_2n_2}\cdots\frac{g_s}{2m}J_{m_kn_k}e^{-\frac{g_s}{8m}\mathrm{Tr}\ J^2}\\
&\qquad\qquad + \cdots\\
&\qquad\qquad + \frac{g_s}{2m}J_{m_1n_1}\cdots\frac{g_s}{2m}J_{m_{k-1}n_{k-1}}\frac{g_s}{2m}(\delta_{mm_k}\delta_{nn_k}-\delta_{mn_k}\delta_{nm_k})e^{-\frac{g_s}{8m}\mathrm{Tr}\ J^2}\\
&\qquad\qquad + \frac{g_s}{2m}J_{mn}\frac{g_s}{2m}J_{m_1n_1}\cdots\frac{g_s}{2m}J_{m_kn_k}e^{-\frac{g_s}{8m}\mathrm{Tr}\ J^2}.
\end{aligned} \tag{C.6}$$

The indices m_i and n_i are contracted in traces as given in the interaction which can be interpreted as forming vertices. The combinatorics can the be interpreted diagrammatically, that one must connect all the legs of the vertices in all possible ways with untwisted and twisted propagators. Each twisted propagator contributes a factor of (-1).

The rules for evaluating a diagram are then:

- Each kind of vertex with multiplicity V_j contributes a factor of $\frac{1}{V_j!}(\frac{g_j}{2jg_s})^{V_j}$.
- Each propagator contributes a factor of $\frac{g_s}{2m}$.
- Each twisted propagator contributes a factor of (-1).
- Each index loop contributes a factor of $M = \frac{2S}{g_s}$.

The combinatorial factor of a diagram can be computed by counting all topologically equivalent ways in which the legs of the vertices can be connected. This has some subtleties, since some diagrams with twisted propagators can actually be planar. To handle this, we make use of the technique described in [33] to draw unoriented diagrams (see also [73,74] for recent work on non-orientable ribbon diagrams in the mathematical literature).

An $\mathbb{RP}^2$ can be drawn in the plane as a disc, where antipodal points on the boundary are identified. $\mathbb{RP}^2$ diagrams can then be drawn on that disc with some propagators going through the cross-cap at the boundary. The propagators going through the cross-cap are twisted propagators, whereas all the others are untwisted propagators.

We can now also draw a planar diagram on the $\mathbb{RP}^2$. If it has more than one vertex, we can push one or several vertices through the cross-cap without destroying the planarity, but all the propagators going through the cross-cap are now twisted propagators. This operation contributes a multiplicative factor of 2^{v-1} to the number of planar diagrams at each order v. See Figure 1 for the enumeration of diagrams with 1 quartic vertex.

Using the relation between p and the number of vertices v_i of valency i according to

$$p = \frac{1}{2}\sum_i i v_i \tag{C.7}$$

the contribution of planar diagrams to the free energy of the $SU(M)$ matrix model is given by

$$\mathcal{F}_0 = \sum_{v=1}^{\infty} \frac{d_v^{(n)}}{v!}(\frac{g_n}{ng_s})^v(\frac{g_s}{m})^p M^l = \sum_{v=1}^{\infty} \frac{d_v^{(n)}}{v!}(\frac{g_n}{ng_s})^v(\frac{g_s}{m})^{\frac{1}{2}nv} M^{2-(1-\frac{n}{2})v}, \tag{C.8}$$

where the sum is over diagrams with v vertices of valence $2n$, $d_v^{(n)}$ is the number of planar diagrams at each order, and l counts the number of boundary loops of the ribbon graph. The propagator for $SU(M)$ theories is twice that of the SO/Sp theories. In the second line we have simplified using (5.39) and (C.7). The number of diagrams of topology S^2 (i.e. planar diagrams) in $SU(M)$ matrix theory with a quartic potential is given by [32]

$$d_v^{(4)} = \frac{(2v-1)!12^v}{(v+2)!} = 2, 36, 1728, 145152, \ldots . \tag{C.9}$$

We are not aware of explicit generating functions for other vertex valences $2n$, but these diagrams can be enumerated by computer to the desired order [75].

If we now include twisted propagators (i.e. enumerate planar diagrams in the SO or Sp matrix models), there is an extra contribution to the set of planar diagrams coming from vertices that have been "flipped", converting untwisted to twisted propagators according to the rule described above.

$$\mathcal{F}_0 = \sum_{v=1}^{\infty} \frac{d_v^{(n)}}{v!}(\frac{g_n}{ng_s})^v(\frac{g_s}{2m})^p M^l = \sum_{v=1}^{\infty} \frac{d_v^{(n)}}{v!}(\frac{g_n}{ng_s})^v(\frac{g_s}{2m})^{\frac{1}{2}nv} M^{2-(1-\frac{n}{2})v}, \tag{C.10}$$

$$d_v^{(4)} = \frac{1}{2}\frac{(2v-1)!24^v}{(v+2)!} = 2, 72, 6912, 1161216, \ldots . \tag{C.11}$$

A similar expression exists for the $\mathbb{RP}^2$ free energy

$$\mathcal{F}_1 = \sum_{v=1}^{\infty} \frac{\widetilde{d_v^{(n)}}}{v!}(\frac{g_n}{ng_s})^v(\frac{g_s}{2m})^p M^{l-1} = \sum_{v=1}^{\infty} \frac{\widetilde{d_v^{(n)}}}{v!}(\frac{g_n}{ng_s})^v(\frac{g_s}{2m})^{\frac{1}{2}nv} M^{1-(1-\frac{n}{2})v}. \tag{C.12}$$

Here the number of diagrams $\widetilde{d_v^{(n)}}$ is counted with a minus sign for each twisted propagator[7]. The relevant planar and $\mathbb{RP}^2$ diagrams were enumerated by computer up to 4 vertices with a quartic potential $W_{\text{tree}} \sim \Phi^4$, to 2 vertices with a sextic potential $W_{\text{tree}} \sim \Phi^6$, and for a single vertex with a potential of degree up to 16. The results are summarized in Table 2 and verify the desired relation:

$$\mathcal{F}_1 = -\frac{1}{2}\frac{\partial \mathcal{F}_0}{\partial M}. \tag{C.13}$$

[7]Gaussian Ensembles are matrix models that have been well-studied in the physics and mathematics literature. The Gaussian Orthogonal and Gaussian Symplectic Ensembles also contain non-oriented ribbon diagrams with twisted propagators, however the propagator is $\langle T^a_b T^c_d \rangle \sim \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$, *i.e.*, there is no relative minus sign between the two terms. This corresponds to counting $\mathbb{RP}^2$ diagrams with a positive sign always. Therefore the free energy of the Gaussian Ensembles differs from that of the Lie Algebra matrix models at sub-leading orders in the genus expansion.

Diagrams with quartic vertices:

Gauge group	Topology	$v = 1$	$v = 2$	$v = 3$	$v = 4$
SU	S^2	$2M^3$	$36M^4$	$1728M^5$	$145152M^6$
SO/Sp	S^2	$2M^3$	$72M^4$	$6912M^5$	$1161216M^6$
SO/Sp	$\mathbb{RP}^2$	$-3M^2$	$-144M^3$	$-17280M^4$	$-3483648M^5$

Diagrams with sextic vertices:

Gauge group	Topology	$v = 1$	$v = 2$
SU	S^2	$5M^4$	$600M^5$
SO/Sp	S^2	$5M^4$	$1200M^6$
SO/Sp	$\mathbb{RP}^2$	$-10M^3$	$-3600M^5$

Table 2: Contribution to the free energy of the $SU/SO/Sp$ matrix models at planar and $\mathbb{RP}^2$ level, for quartic and sextic potentials. The first few terms in the perturbative expansion are listed, corresponding to the number of diagrams with increasing number of vertices (equivalently loops).

References

[1] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," `hep-th/0206255`.

[2] R. Dijkgraaf and C. Vafa, "On geometry and matrix models," `hep-th/0207106`.

[3] R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," `hep-th/0208048`.

[4] N. Dorey, T. J. Hollowood, S. Prem Kumar, and A. Sinkovics, "Exact superpotentials from matrix models," `hep-th/0209089`.

[5] N. Dorey, T. J. Hollowood, S. P. Kumar, and A. Sinkovics, "Massive vacua of $N = 1^*$ theory and S-duality from matrix models," `hep-th/0209099`.

[6] L. Chekhov and A. Mironov, "Matrix models vs. Seiberg-Witten/Whitham theories," `hep-th/0209085`.

[7] N. Dorey, T. J. Hollowood, and S. P. Kumar, "S-duality of the Leigh-Strassler deformation via matrix models," `hep-th/0210239`.

[8] F. Ferrari, "Quantum parameter space and double scaling limits in $N = 1$ super Yang-Mills theory," hep-th/0211069.

[9] D. Berenstein, "Quantum moduli spaces from matrix models," hep-th/0210183.

[10] H. Fuji and Y. Ookouchi, "Comments on effective superpotentials via matrix models," hep-th/0210148.

[11] R. Gopakumar, "$N = 1$ theories and a geometric master field," hep-th/0211100.

[12] Y. Demasure and R. A. Janik, "Effective matter superpotentials from Wishart random matrices," hep-th/0211082.

[13] R. Argurio, V. L. Campos, G. Ferretti, and R. Heise, "Exact superpotentials for theories with flavors via a matrix integral," hep-th/0210291.

[14] J. McGreevy, "Adding flavor to Dijkgraaf-Vafa," hep-th/0211009.

[15] H. Suzuki, "Perturbative derivation of exact superpotential for meson fields from matrix theories with one flavour," hep-th/0211052.

[16] I. Bena and R. Roiban, "Exact superpotentials in $N = 1$ theories with flavor and their matrix model formulation," hep-th/0211075.

[17] A. Gorsky, "Konishi anomaly and $N = 1$ effective superpotentials from matrix models," hep-th/0210281.

[18] S. Naculich, H. Schnitzer, and N. Wyllard, "The $\mathcal{N} = 2$ $U(N)$ gauge theory prepotential and periods from a perturbative matrix model calculation," hep-th/0211123.

[19] Y. Tachikawa, "Derivation of the Konishi anomaly relation from Dijkgraaf- Vafa with (bi-)fundamental matters," hep-th/0211189.

[20] R. Dijkgraaf, A. Neitzke, and C. Vafa, "Large N strong coupling dynamics in non-supersymmetric orbifold field theories," hep-th/0211194.

[21] A. Klemm, M. Marino, and S. Theisen, "Gravitational corrections in supersymmetric gauge theory and matrix models," hep-th/0211216.

[22] R. Dijkgraaf, A. Sinkovics, and M. Temurhan, "Matrix models and gravitational corrections," hep-th/0211241.

[23] B. Feng, "Seiberg duality in matrix models," hep-th/0211202.

[24] B. Feng and Y.-H. He, "Seiberg duality in matrix models II," hep-th/0211234.

[25] R. Argurio, V. L. Campos, G. Ferretti, and R. Heise, "Baryonic corrections to superpotentials from perturbation theory," hep-th/0211249.

[26] S. Naculich, H. Schnitzer, and N. Wyllard, "Matrix model approach to the $\mathcal{N} = 2$ $U(N)$ gauge theory with matter in the fundamental representation," hep-th/0211254.

[27] F. Ferrari, "On exact superpotentials in confining vacua," hep-th/0210135.

[28] R. Dijkgraaf, M. T. Grisaru, C. S. Lam, C. Vafa, and D. Zanon, "Perturbative computation of glueball superpotentials," hep-th/0211017.

[29] F. Cachazo, M. R. Douglas, N. Seiberg, and E. Witten, "Chiral rings and anomalies in supersymmetric gauge theory," hep-th/0211170.

[30] J. Ambjorn, L. Chekhov, C. F. Kristjansen, and Y. Makeenko, "Matrix model calculations beyond the spherical limit," *Nucl. Phys.* **B404** (1993) 127–172, hep-th/9302014.

[31] G. Akemann, "Higher genus correlators for the Hermitian matrix model with multiple cuts," *Nucl. Phys.* **B482** (1996) 403–430, hep-th/9606004.

[32] E. Brezin, C. Itzykson, G. Parisi, and J. B. Zuber, "Planar diagrams," *Commun. Math. Phys.* **59** (1978) 35.

[33] G. M. Cicuta, "Topological expansion for $SO(N)$ and $Sp(2N)$ gauge theories," *Nuovo Cim. Lett.* **35** (1982) 87.

[34] N. Dorey, "An elliptic superpotential for softly broken $N = 4$ supersymmetric Yang-Mills theory," *JHEP* **07** (1999) 021, hep-th/9906011.

[35] H. Ita, H. Nieder, and Y. Oz, "Perturbative computation of glueball superpotentials for $SO(N)$ and $USp(N)$," hep-th/0211261.

[36] F. Cachazo, K. A. Intriligator, and C. Vafa, "A large N duality via a geometric transition," *Nucl. Phys.* **B603** (2001) 3–41, hep-th/0103067.

[37] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory," *Nucl. Phys.* **B426** (1994) 19–52, hep-th/9407087.

[38] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD," *Nucl. Phys.* **B431** (1994) 484–550, hep-th/9408099.

[39] W. Lerche, "Introduction to Seiberg-Witten theory and its stringy origin," *Nucl. Phys. Proc. Suppl.* **55B** (1997) 83–117, hep-th/9611190.

[40] L. Alvarez-Gaume and S. F. Hassan, "Introduction to S-duality in $N = 2$ supersymmetric gauge theories: A pedagogical review of the work of Seiberg and Witten," *Fortsch. Phys.* **45** (1997) 159–236, hep-th/9701069.

[41] U. H. Danielsson and B. Sundborg, "The moduli space and monodromies of $N = 2$ supersymmetric $SO(2r + 1)$ Yang-Mills theory," *Phys. Lett.* **B358** (1995) 273–280, hep-th/9504102.

[42] A. Brandhuber and K. Landsteiner, "On the monodromies of $N = 2$ supersymmetric Yang-Mills theory with gauge group $SO(2n)$," *Phys. Lett.* **B358** (1995) 73–80, hep-th/9507008.

[43] P. C. Argyres and A. D. Shapere, "The vacuum structure of $N = 2$ superQCD with classical gauge groups," *Nucl. Phys.* **B461** (1996) 437–459, hep-th/9509175.

[44] F. Cachazo and C. Vafa, "$N = 1$ and $N = 2$ geometry from fluxes," hep-th/0206017.

[45] R. Donagi and E. Witten, "Supersymmetric Yang-Mills theory and integrable systems," *Nucl. Phys.* **B460** (1996) 299–334, hep-th/9510101.

[46] N. Dorey and S. P. Kumar, "Softly-broken $\mathcal{N} = 4$ supersymmetry in the large-N limit," *JHEP* **02** (2000) 006, hep-th/0001103.

[47] N. Seiberg and E. Witten, "Gauge dynamics and compactification to three dimensions," in *The mathematical beauty of physics: A memorial volume for Claude Itzykson.* (Saclay, France, June 5-7, 1996), J. M. Drouffe and J. B. Zuber, eds. World Scientific, Singapore, 1996. `hep-th/9607163`.

[48] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov, and A. Morozov, "Integrability and Seiberg-Witten exact solution," *Phys. Lett.* **B355** (1995) 466–474, `hep-th/9505035`.

[49] E. J. Martinec and N. P. Warner, "Integrable systems and supersymmetric gauge theory," *Nucl. Phys.* **B459** (1996) 97–112, `hep-th/9509161`.

[50] E. J. Martinec, "Integrable structures in supersymmetric gauge and string theory," *Phys. Lett.* **B367** (1996) 91–96, `hep-th/9510204`.

[51] E. D'Hoker and D. H. Phong, "Calogero-Moser systems in $SU(N)$ Seiberg-Witten theory," *Nucl. Phys.* **B513** (1998) 405–444, `hep-th/9709053`.

[52] E. D'Hoker and D. H. Phong, "Calogero-Moser Lax pairs with spectral parameter for general Lie algebras," *Nucl. Phys.* **B530** (1998) 537–610, `hep-th/9804124`.

[53] E. D'Hoker and D. H. Phong, "Calogero-Moser and Toda systems for twisted and untwisted affine Lie algebras," *Nucl. Phys.* **B530** (1998) 611–640, `hep-th/9804125`.

[54] R. Corrado. Work in progress.

[55] J. D. Edelstein, K. Oh, and R. Tatar, "Orientifold, geometric transition and large N duality for SO/Sp gauge theories," *JHEP* **05** (2001) 009, `hep-th/0104037`.

[56] S. Sinha and C. Vafa, "SO and Sp Chern-Simons at large N," `hep-th/0012136`.

[57] B. Acharya, M. Aganagic, K. Hori, and C. Vafa, "Orientifolds, mirror symmetry and superpotentials," `hep-th/0202208`.

[58] H. Fuji and Y. Ookouchi, "Confining phase superpotentials for SO/Sp gauge theories via geometric transition," `hep-th/0205301`.

[59] J. Gomis, "On SUSY breaking and χSB from string duals," *Nucl. Phys.* **B624** (2002) 181–199, `hep-th/0111060`.

[60] C. Vafa, "Superstrings and topological strings at large N," *J. Math. Phys.* **42** (2001) 2798–2817, hep-th/0008142.

[61] T. R. Taylor and C. Vafa, "RR flux on Calabi-Yau and partial supersymmetry breaking," *Phys. Lett.* **B474** (2000) 130–137, hep-th/9912152.

[62] K. Becker and M. Becker, "M-Theory on eight-manifolds," *Nucl. Phys.* **B477** (1996) 155–167, hep-th/9605053.

[63] S. Gukov, C. Vafa, and E. Witten, "CFT's from Calabi-Yau four-folds," *Nucl. Phys.* **B584** (2000) 69–108, hep-th/9906070.

[64] J. Polchinski and A. Strominger, "New vacua for type II string theory," *Phys. Lett.* **B388** (1996) 736–742, hep-th/9510227.

[65] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, "2-D gravity and random matrices," *Phys. Rept.* **254** (1995) 1–133, hep-th/9306153.

[66] R. Dijkgraaf, S. Gukov, V. A. Kazakov, and C. Vafa, "Perturbative analysis of gauged matrix models," hep-th/0210238.

[67] R. L. Mkrtchian, "The equivalence of $Sp(2N)$ and $SO(2N)$ gauge theories," *Phys. Lett.* **B105** (1981) 174–176.

[68] P. Cvitanovic and A. D. Kennedy, "Spinors in negative dimensions," *Phys. Scripta* **26** (1982) 5.

[69] H. Ooguri and C. Vafa, "Worldsheet derivation of a large N duality," *Nucl. Phys.* **B641** (2002) 3–34, hep-th/0205297.

[70] E. Witten, "Constraints on supersymmetry breaking," *Nucl. Phys.* **B202** (1982) 253.

[71] E. Witten, "Toroidal compactification without vector structure," *JHEP* **02** (1998) 006, hep-th/9712028.

[72] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," *Commun. Math. Phys.* **165** (1994) 311–428, hep-th/9309140.

[73] M. Mulase and A. Waldron, “Duality of orthogonal and symplectic matrix integrals and quaternionic Feynman graphs,” math-ph/0206011.

[74] M. Mulase and J. T. Yu, “Non-commutative matrix integrals and representation varieties of surface groups in a finite group,” math.QA/0211127.

[75] K. D. Kennaway. C code to enumerate planar and $\mathbb{RP}^2$ Feynman diagrams (included with hep-th/0211291 source file).